Polyhedral model of fibers of periods map

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October 31, 2014

[1] supported by RFBR 13-01-00115 & 'Modern Problems of theoretical Math' RAS Programme

Motivation

Solutions of several rational approximation problems in C-norm

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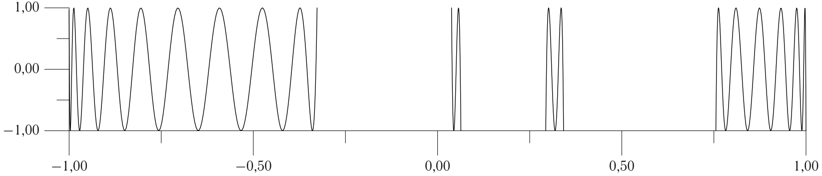

Figure: Degree 29 Chebyshev polynomial on four segments

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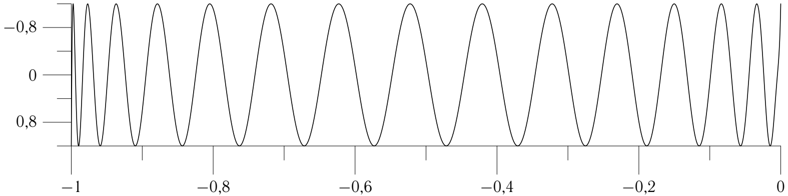

Figure: Degree $n = 31$ Optimal Stability Polynomial for RK method of accuracy degree $p = 3$

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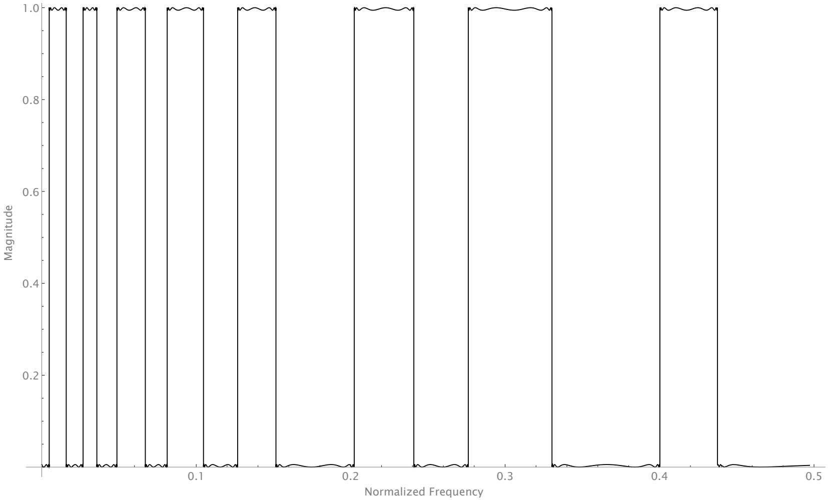


Figure: Optimal gain-frequency characteristic of multiband electrical filter, degree $n = 326$, $g = 16$ (Computed by Sergei Lyamaev)

Motivation: continued

Oscillatory behaviuor of solutions a.k.a. equiripple property or Chebyshev's alternation principle.

Chebyshev representation of polynomials

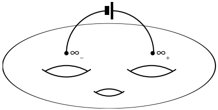
$$P(x) = \pm \cos(ni \int_{(e,0)}^{(x,w)} d\eta_M)$$

in terms of associated hyperelliptic curve $M(\mathrm{E})$

$$w^2 = \prod_{s=1}^{2g+2} (x - e_s), \qquad (x, w) \in \mathbb{C}^2, \qquad \mathrm{E} = \{e_s\},$$

and distinguished differential $d\eta_M$ on it:
simple poles at infinity with residues ± 1 and purely imaginary periods:

$$d\eta_M = (x^g + \dots)\frac{dx}{w}$$



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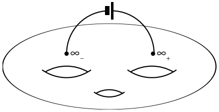

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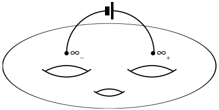

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Motivation: continued

Example:

Sphere $g = 0$ gives Chebyshëv polynomials (1853);
Tori $g = 1$ give a family of Zolotarëv polynomials (1868).

Abelian equations

$$\int_C d\eta_M \in \frac{2\pi i}{n}\mathbb{Z}, \quad \forall C \in H_1(M, \mathbb{Z}).$$

Q: How to conserve Abelian Eqs on a variable curve M?
A: Move along fibers of period map.

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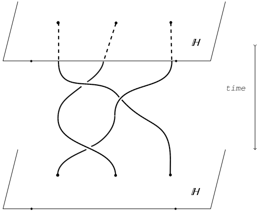
Moduli spaces of real hyperelliptic curves

Consider moduli space $\mathcal{H}$ of real hyperelliptic curves with a marked point ∞ ($\neq$ branchpoint) on an oriented real oval. Curves with fixed topological invariants: the number k of real ovals and the genus g make up a component $\mathcal{H}_g^k$. Half of the symmetric branching divisor $\mathrm{E} = \bar{\mathrm{E}}$: $2k$ real points and $g-k+1$ points of the upper half plane (modulo translations and dilatations).



$$\mathcal{H}_g^k := \mathbb{H}^{g-k+1} \backslash \{diagonals\}/permutations \times \triangle_{2k-2}$$

$\dim_{\mathbb{R}} \mathcal{H}_g^k = 2g$;
$\pi_1(\mathcal{H}_g^k) = Br_{g-k+1}$ (braids on $g-k+1$ strings).

Period mapping

Locally we can define the period map $\mathcal{H}_g^k \to \mathbb{R}^g$ as follows: Given a basis C_j in $H_1(M, \mathbb{Z})$,

$$\Pi_j(\mathrm{E}) = -i \int_{C_j} d\eta_{M(E)}, \qquad j = 1, \dots, 2g.$$

Globally the map is not correctly defined because of the monodromy of Gauss-Manin connection: braids entangle the basic odd cycles C_j^- (Burau representation).
However, the period map Π is well defined on the universal cover of the moduli space: $\Pi : \quad \tilde{\mathcal{H}}_g^k \simeq \mathbb{R}^{2g} \to \mathbb{R}^g$.

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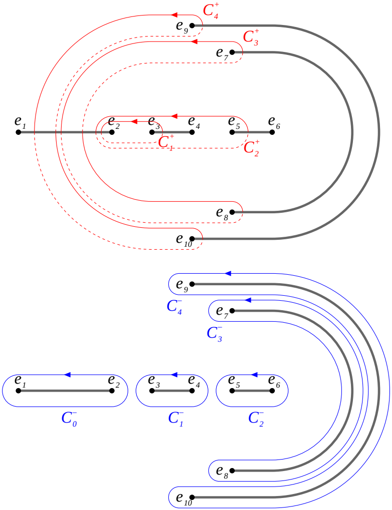
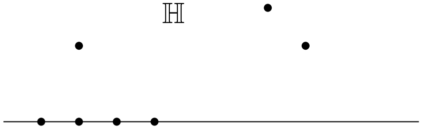
Digression: Even and Odd Cycles

Simplification due to mirror symetry $\bar{J}(x, w) = (\bar{x}, \bar{w})$: cycles are split into even/odd: $\bar{J}C = \pm C$. Real differential $d\eta_M$ has trivial periods along all even cycles due to its normalization.



Labyrinth model of the moduli space universal cover



A point $E \in \mathcal{H}_4^2$ is lifted to the universal cover by choosing the labyrinth that accompanies it.

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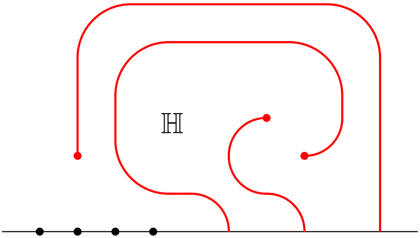

Labyrinth of a point $\mathrm{E} \in \mathcal{H}$ gives a natural basis in odd 1-homologies of the curve $M(\mathrm{E})$. Fundamental group of the base (braids) acts on labyrinths as MCG of punctured half plane.

Period mapping F.A.Q.

Natural questions about period mapping arise:

- Are fibers of Π smooth?
- How many rational fibers are there? (they parametrize solutions to extremal problems)
- What is the range of Π?
- How Π interacts with braids? Are there fixed fibers?
- What is the global topology of a fiber? Connected?

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CONJECTURE(2001): Component of a fiber $=$ g-cell.

Pictorial representation of curves

Let us fix a curve $M \in \mathcal{H}$. Due to normalization of distinguished differential, the function

$$W(x) := |\mathrm{Re} \int_{(e,0)}^{(x,w)} d\eta_M|$$

- is single valued on the plane,
- harmonic outside its zero set (containing all branchpoints)
- has logarithmic pole at infinity.
- it's level sets are the leaves of the foliation $d\eta_M^2 < 0$ on the sphere.

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Construction of a graph $\Gamma(M) := \Gamma_{|} \cup \Gamma_{-}$.

- $\Gamma_{|}$ is zero set of $W(x)$, not oriented;
- Γ_{-} are all segments of the horizontal foliation $d\eta_M^2 > 0$ oriented with respect to the growth of $W(x)$ and connecting the finite critical points of the foliation to other such points or to zeroset of W.
- Each edge is equipped with its length in the metric $|d\eta_M|$.
- The vertices of the graph are the finite points of the divisor $(d\eta_M{}^2)$ and their projections to the vertical component along the horizontal foliation.

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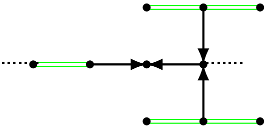
Example of associated graph



Remarks:

- The multiplicity of V in divisor of $(d\eta_M)^2$ equals to $ord(V) := d_{|}(V) + 2d_{in}(V) - 2$. Hence, combinatorics of the graph $\Gamma(M)$ gives topological invariants g, k of the curve M.
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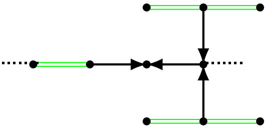

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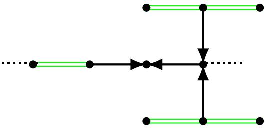

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Axiomatic description of graphs

Properties of associated graph:

1. Γ is a tree with horizontal symmety axis (Topology)
2. Outcoming horizontal edges are separated (Topology)
3. $W(V) = 0$ iff V is on the vertical part of the graph (Width normalization)
4. If $\mathrm{ord}(V) = 0$ then $V \in \Gamma_{\_} \cap \Gamma_{|}$ (Minimal vertices)
5. The lengths of all vertical edges is π. (Height normalization)

Theorem

Each weighted graph satisfying the above properties 1-5 stems from a unique curve $M \in \mathcal{H}$.

Proof hint: The Riemann surface may be glued from a finite number of stripes in a way determined by combinatorics and weights of graph.

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Coordinate space of a graph

The weights of the admissible graph have obvious linear restrictions. They fill out a convex polyhedron $\mathcal{A}[\Gamma]$: symplex $\{H_s\}\times$ cone $\{W_j\}$ of dimension at most $2g$.

1. $\sum_s H_s = \pi$ symplex
2. *if* $V_1 \longrightarrow V_2$ *then* $W_1 < W_2$, $V_* \in \Gamma_{\_}$ cone

Example

$g = 2$;
$k = 1$;
$dim\mathcal{A}[\Gamma] = 2g = 4$;
$\mathcal{A}[\Gamma] = \{2(H_1 + H_2) + H_3 = \pi\}\times$ $\{0 < W_1 < W_2\}$

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Space $\mathcal{A}[\Gamma]$ is real analytically embedded to the moduli space $\mathcal{H}$.

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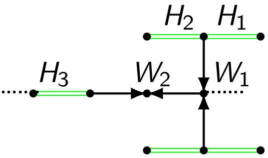

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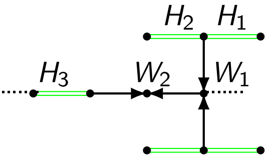

Example

$g = 2$;
$k = 1$;
$dim\mathcal{A}[\Gamma] = 2g = 4$;
$\mathcal{A}[\Gamma] = \{2(H_1 + H_2) + H_3 = \pi\}\times$
$\{0 < W_1 < W_2\}$

Theorem

Space $\mathcal{A}[\Gamma]$ is real analytically embedded to the moduli space $\mathcal{H}$.

Polyhedral model of Moduli space

We've built a cellular decomposition of the moduli space, cells are encoded by admissible types of trees. It's polyhedral model is made in two steps:

Polyhedral model of Moduli space

A List all admissible graphs Γ with given g, k with full $dim\mathcal{A}[\Gamma] = 2g$

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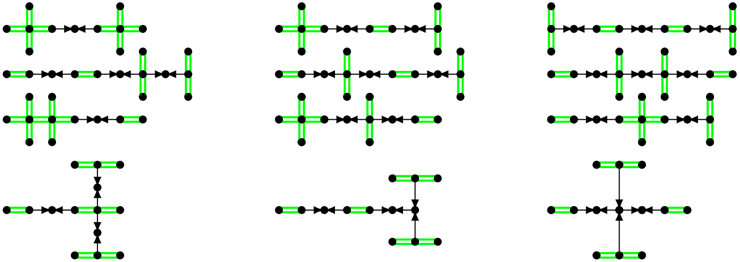
EXAMPLE: 20 codimension zero cells in the space $\mathcal{H}_3^2$ (up to symmetry)



Polyhedral model of Moduli space

A List all admissible graphs Γ with given g, k with full $dim\mathcal{A}[\Gamma] = 2g$

B Glue faces of polyhedra with the help of Neighboring relations

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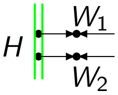
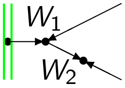


when $H \to 0$ goes to



if $W_1 < W_2$.

Polyhedral model of Universal cover of moduli space

For all full dimensional polyhedra $\mathcal{A}[\Gamma]$ there is a canonical lift to $\tilde{\mathcal{H}}$ by attaching a labyrinth not intersecting the graph Γ.

Model of the universal cover:

$$\tilde{\mathcal{H}}_g^k = \cup\, \beta \cdot \mathcal{A}[\Gamma],$$

braids $\beta \in Br_{g-k+1}$, and graphs with given $g(\Gamma)$ and $k(\Gamma)$.

Glueing rules:

A polyhedron $\beta_1 \cdot \mathcal{A}[\Gamma_1]$ attached to $\beta_2 \cdot \mathcal{A}[\Gamma_2]$ along their common face Γ_{12} iff $\beta_1 = \beta_{12}\beta_2$, where the braid β_{12} maps the labyrinth inherited by Γ_{12} from Γ_1 to the labyrinth inherited from Γ_2.

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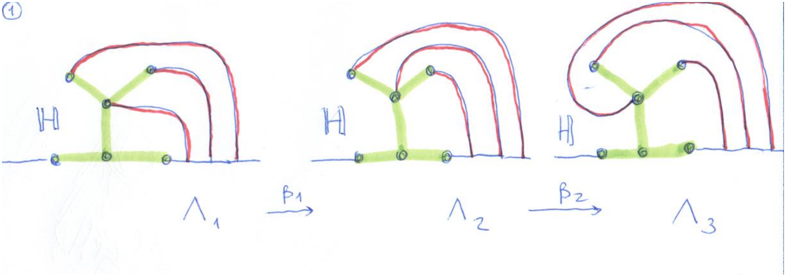
Exceptional graphs = graphs with nonhanging branchpoints in upper halfplane.



Graph from $\mathcal{H}_3^1$ admitting three labyrinths transformed by braids generators.

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Periods map

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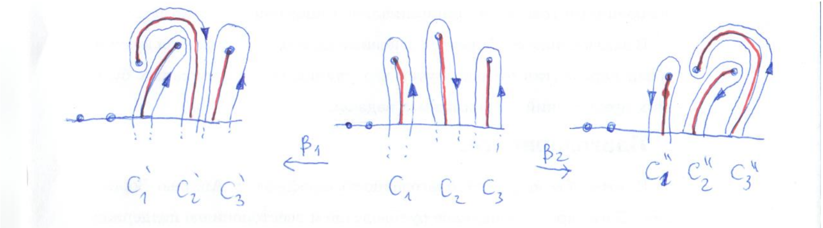


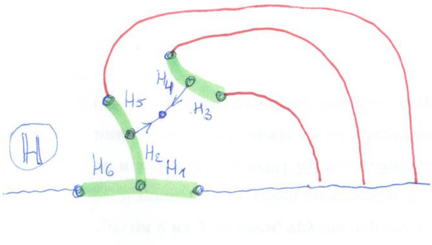
The holonomy of the connection looks like Burau representation of braids:

$$\beta_1 \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_2 \\ 2C_2 - C_1 \\ C_3 \end{pmatrix} \qquad \beta_2 \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_3 \\ 2C_3 - C_1 \end{pmatrix}$$

Periods map

Period map is a liner function in local coordinates (heights) of the cell. It is easy to compute it for the associated labyrinth of a graph:

EXAMPLE:



$$\Pi(\Lambda) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} =$$

$$= \begin{pmatrix} H_1 + H_2 + H_3 \\ H_1 + H_2 + H_3 + H_4 \\ H_1 + H_2 + H_3 + 2H_4 + H_5 \end{pmatrix}$$

Value of $\Pi(\Lambda)$ lies in a symplex Δ_3 $0 < h_1 < h_2 < h_3 < \pi$

Polyhedral model of fibers of Periods map

Let us introduce a intersection $\mathcal{A}[\Gamma, h]$ of the polyhedron $\mathcal{A}[\Gamma] \subset \mathcal{H}$ and a fiber $\Pi^{-1}(h)$, h from symplex Δ_g. This is a polyhedron of dimension g.

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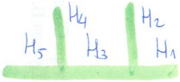
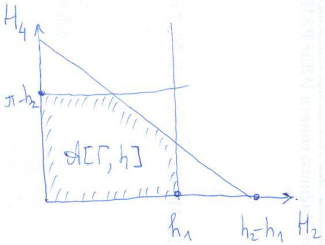
EXAMPLE Polygon $\mathcal{A}[\Gamma, h]$ for $\Gamma = \Gamma_{\text{I}}$ from $\mathcal{H}_2^1$ (this space contains 9 full dimension cells)





Fix the periods:

$H_1 + H_2 = h_1$

$H_1 + 2H_2 + H_3 + H_4 = h_2$

$H_1 + 2H_2 + H_3 + 2H_4 + H_5 = \pi$

(normalization)

Positive coordinates H_2, H_4 in the polygon satisfy

$H_2 < h_1$

$H_4 < \pi - h_2$

$H_2 + H_4 < h_2 - h_1$

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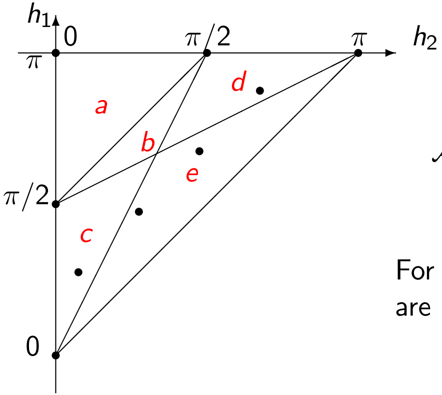
Phase diagram for the space $\mathcal{H}_2^1$



$$\mathcal{A}[\Gamma, h] = \begin{cases} \textit{rectangle} & h \in a \\ \textit{pentagon} & h \in b \\ \textit{trapezoid} & h \in c, d \\ \textit{triangle} & h \in e \end{cases}$$

For other graphs the polygons $\mathcal{A}[\Gamma, h]$ are half-stripes or quadrants or empty.

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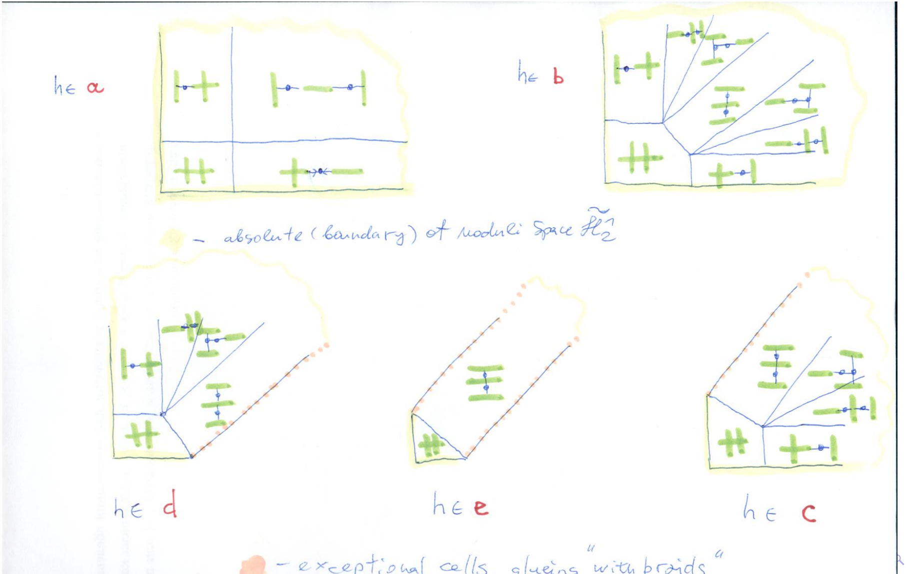
Assembling the Period map fiber from cells:

$$\Pi^{-1}(h) = \cup\, \beta \cdot \mathcal{A}[\Gamma, \beta^{-1} \cdot h],$$

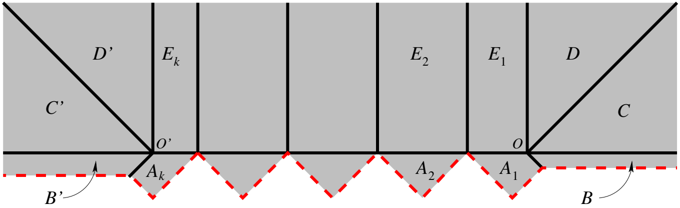
braids: $\beta^{-1} \cdot h \in \Delta_g$; graphs: with given topological invariants g, k.

Assembly for fibers of the space $\tilde{\mathcal{H}}_2^1$

Fiber $\Pi^{-1}(h)$ with h from the above phase diagram:



The above fiber in 'e' phase assembled



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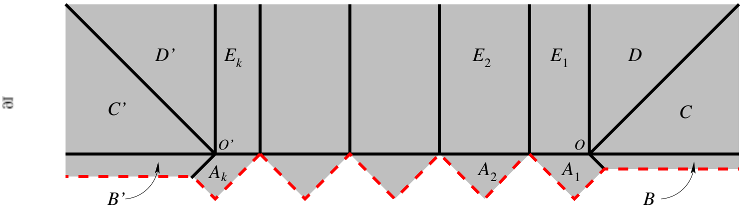

This is Sasha Zvonkin's picture of a fiber.

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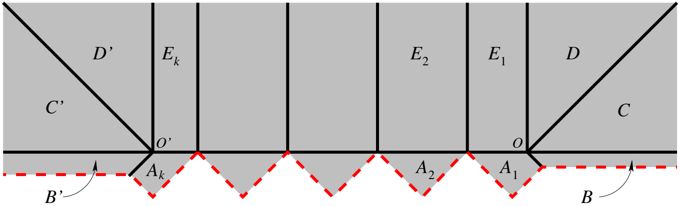

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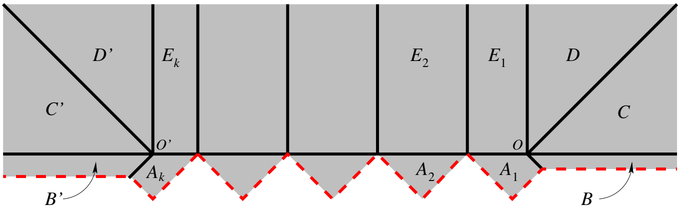

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I thank everyone for the patience!